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LETTER TO THE EDITOR

A disorder solution of the general 8-vertex model

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Abstract. A disorder solution for the general 8-vertex model is found by applying a procedure recently developed by Baxter. The solution is valid when an adequate relation between the vertex weights of the model is satisfied.

Very recently a number of papers have appeared that deal with disorder solutions of particular two- and three-dimensional lattice models (Rujan 1982, 1984, Peschel and Rys 1982, Baxter 1984, Jaekel and Maillard 1985a,b,c, Wu 1985, Chao and Wu 1985, Hansel *et al* 1985).

The disorder solutions are very useful for clarifying the phase diagrams of anisotropic models and also imply constraints on the analytical behaviour of the partition function of these models.

Besides, it has been shown very recently that it is possible to obtain information in the vicinity of disorder solutions through a new type of perturbative expansion (Hansel *et al* 1985).

All the methods applied for obtaining disorder solutions rely on the same mechanism: a certain local decoupling of the degrees of freedom of the model, which results in an effective reduction of dimensionality for the lattice system. In the works of Jaekel and Maillard (1985b), Wu (1985) and Baxter (1984) such a property is provided by a simple local condition imposed on the Boltzmann weights of the elementary cell generating the lattice. However, although very similar, the procedure developed by Baxter is not equivalent to the procedure used by Jaekel and Maillard, and Wu. Both methods have different realms of applications. In Baxter's work (Baxter 1984) the general IRF model (Baxter 1980) on the square lattice is analysed, and a condition is given for the model to have a disorder solution.

In this letter this condition is applied to the 8-vertex model with horizontal and vertical electric fields (general 8-vertex model), which, when formulated in terms of Ising spins, is a special case of the spin-$\frac{1}{2}$ IRF model (Baxter 1982a). A particular case of the 8-vertex model, when one of the electric fields is null, has been treated by Rujan (1982) and Peschel and Rys (1982). By using the transfer matrix method, and a correspondence between vertex models and one-dimensional quantum theories of the *xyz* type, respectively, a disorder solution is found by these authors. In the present letter this result is generalised to the case when horizontal and vertical electric fields are present. The square lattice is drawn diagonally, as in figure 1, and periodic boundary conditions are imposed.

With each site i a spin σ_i, which takes the values $+1$ and -1, is associated, and to each face a Boltzmann weight factor $W(\sigma_i, \sigma_j, \sigma_k, \sigma_l)$ is assigned. Here i, j, k, l indicate the four sites round the face, arranged anticlockwise as shown in figure 1. The partition function is

$$Z=\sum_{(\sigma)}\prod_{(i,j,k,l)} W(\sigma_i,\sigma_j,\sigma_k,\sigma_l) \tag{1}$$

where the sum is over all values of all spins, and the product is over all faces of the lattice.

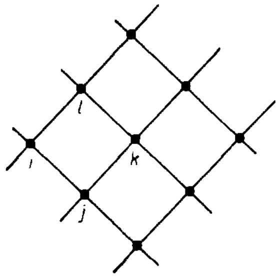

Figure 1. The square lattice, drawn diagonally, showing a face i, j, k, l.

For the general 8-vertex model the Boltzmann weights are given by

$$\begin{aligned} W(\sigma_i,\sigma_j,\sigma_k,\sigma_l) \\ &=\exp(K_1\sigma_i\sigma_j+K_2\sigma_i\sigma_l+K_3\sigma_l\sigma_k+K_4\sigma_j\sigma_k \\ &\quad+K_5\sigma_i\sigma_k+K_6\sigma_l\sigma_j+K_7\sigma_i\sigma_j\sigma_k\sigma_l) \end{aligned} \tag{2}$$

and the vertex weights are

$$\begin{aligned} &\omega_1=W(++++) && \omega_5=W(++-+) \\ &\omega_2=W(+-+-) && \omega_6=W(-+++) \\ &\omega_3=W(+--+) && \omega_7=W(+++-) \\ &\omega_4=W(++--) && \omega_8=W(+-++). \end{aligned} \tag{3}$$

The condition given by Baxter for the model to have a disorder solution is as follows: if there exists κ, $\varphi(\sigma)$ and non-negative functions $f(\sigma_1,\sigma_2)$ and $g(\sigma_1,\sigma_2)$ such that

$$\sum_{\sigma_2'} W(\sigma_1,\sigma_2,\sigma_3,\sigma_2')g(\sigma_1,\sigma_2')f(\sigma_2',\sigma_3)=\kappa\varphi(\sigma_1)f(\sigma_1,\sigma_2)g(\sigma_2,\sigma_3)/\varphi(\sigma_3) \tag{4}$$

for all values of σ_1, σ_2, σ_3, then the partition function per site is given by κ. The four spins σ_1, σ_2, σ_3, σ_2' are located on the four sites round a face of the lattice, as indicated in figure 2. When the condition (4) is satisfied, the maximal right eigenvector of the transfer matrix is a direct product of factors, each factor corresponding to an edge of the lattice.

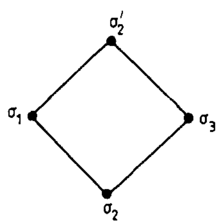

Figure 2. The four spins σ_1, σ_2, σ_3 and σ_2' round a face of the lattice.

Since the Boltzmann weights of the 8-vertex model are invariant under spin reversal, we impose on the functions φ, f and g the same symmetry property:

$$g(\sigma_1,\sigma_2)=g(-\sigma_1,-\sigma_2) \qquad \varphi(\sigma)=\varphi(-\sigma)$$
$$f(\sigma_1,\sigma_2)=f(-\sigma_1,-\sigma_2). \tag{5}$$

Hence we have

$$f(\sigma_1,\sigma_2)=\exp(L_1\sigma_1\sigma_2) \qquad \varphi(\sigma)\equiv 1$$
$$g(\sigma_1,\sigma_2)=\exp(L_2\sigma_1\sigma_2) \tag{6}$$

where L_1 and L_2 are real.

Therefore the condition (4) can be written as

$$\sum_{\sigma_2'} W(\sigma_1,\sigma_2,\sigma_3,\sigma_2')\exp[L_1(\sigma_2'\sigma_3-\sigma_1\sigma_2)+L_2(\sigma_1\sigma_2'-\sigma_2\sigma_3)]=\kappa \tag{7}$$

for all values of $\sigma_1,\sigma_2,\sigma_3$.

Taking into account (2), the product of $W(\sigma_1,\sigma_2,\sigma_3,\sigma_2')$ by the factor $\exp[L_1(\sigma_2'\sigma_3-\sigma_1\sigma_2)+L_2(\sigma_1\sigma_2'-\sigma_2\sigma_3)]$ is equivalent to redefining the parameters K_1, K_2, K_3 and K_4. Moreover, if we change the Boltzmann weights as follows:

$$\hat{W}(\sigma_1,\sigma_2,\sigma_3,\sigma_2')=W(\sigma_1,\sigma_2,\sigma_3,\sigma_2')\exp[L_1(\sigma_2'\sigma_3-\sigma_1\sigma_2)+L_2(\sigma_1\sigma_2'-\sigma_2\sigma_3)] \tag{8}$$

the partition function is left unchanged, because the incorporated factors cancel two by two. In consequence, without loss of generality we can take $L_1=L_2=0$, and equation (7) now becomes

$$\sum_{\sigma_2'} W(\sigma_1,\sigma_2,\sigma_3,\sigma_2')=\kappa \tag{9}$$

for all values of $\sigma_1,\sigma_2,\sigma_3$

Taking all the possible values of σ_1, σ_2 and σ_3, and taking into account (3) and the symmetry $W(-\sigma_1,-\sigma_2,-\sigma_3,-\sigma_2')=W(\sigma_1,\sigma_2,\sigma_3,\sigma_2')$, only four independent equations are obtained from (9):

$$\omega_1+\omega_7=\kappa \qquad \omega_2+\omega_8=\kappa$$
$$\omega_4+\omega_5=\kappa \qquad \omega_3+\omega_6=\kappa \tag{10}$$

from which the following relations for the weights are deduced:

$$\omega_5=\omega_1+\omega_7-\omega_4 \qquad \omega_6=\omega_1+\omega_7-\omega_3$$
$$\omega_8=\omega_1+\omega_7-\omega_2. \tag{11}$$

However, the partition function of the 8-vertex model depends on the weights $\omega_1,\omega_2,\omega_3,\omega_4$ and on the effective weights $\tilde{\omega}_5=(\omega_5\omega_6)^{1/2}$ and $\tilde{\omega}_7=(\omega_7\omega_8)^{1/2}$ (Fan and Wu 1970). Using (11) the effective weights $\tilde{\omega}_5$ and $\tilde{\omega}_7$ are given by

$$\tilde{\omega}_5=[(\omega_1+\omega_7-\omega_4)(\omega_1+\omega_7-\omega_3)]^{1/2} \tag{12a}$$
$$\tilde{\omega}_7=[\omega_7(\omega_1+\omega_7-\omega_2)]^{1/2}. \tag{12b}$$

From (12*b*) ω_7 can be expressed in terms of $\tilde{\omega}_7$:

$$\omega_7=\tfrac{1}{2}\{\omega_2-\omega_1+[(\omega_2-\omega_1)^2+4\tilde{\omega}_7^2]^{1/2}\} \tag{13}$$

and then replaced in (12a), resulting in

$$\tilde{\omega}_5 = [(\alpha - \omega_4)(\alpha - \omega_3)]^{1/2} \tag{14}$$

where

$$\alpha = \tfrac{1}{2}\{\omega_1 + \omega_2 + [(\omega_1 - \omega_2)^2 + 4\tilde{\omega}_7^2]^{1/2}\}.$$

Moreover, for all the weights to be non-negative, the following inequalities must be satisfied:

$$\alpha - \omega_4 \geqslant 0 \qquad \alpha - \omega_3 \geqslant 0. \tag{15}$$

Therefore there are five independent weights ω_1, ω_2, ω_3, ω_4 and $\tilde{\omega}_7$, and one of the weights ($\tilde{\omega}_5$ in this case) is determined through equation (14).

From (10) and (13), the partition function per site is given by

$$\kappa = \tfrac{1}{2}\{\omega_1 + \omega_2 + [(\omega_1 - \omega_2)^2 + 4\tilde{\omega}_7^2]^{1/2}\}. \tag{16}$$

Other disorder solutions can be obtained from (14) and (16) by interchanging the weights ω_i in accordance with the symmetry relations verified for the 8-vertex model (Fan and Wu 1970).

Baxter has also shown that if the additional relation

$$\sum_{\sigma_2'} W(\sigma_1, \sigma_2', \sigma_3, \sigma_2)\tilde{f}(\sigma_1, \sigma_2')\tilde{g}(\sigma_2', \sigma_3) = \kappa\tilde{\varphi}(\sigma_1)\tilde{g}(\sigma_1, \sigma_2)\tilde{f}(\sigma_2, \sigma_3)/\tilde{\varphi}(\sigma_3) \tag{17}$$

is satisfied, where $\tilde{\varphi}$, $\tilde{f}$ and $\tilde{g}$ are non-negative functions, then the left eigenvector of the transfer matrix is also a direct product of factors, each factor corresponding to an edge of the lattice. Moreover the correlations within a horizontal row are those of a one-dimensional nearest-neighbour model and therefore decay exponentially with distance. Then there can be no long-range order, at least in the horizontal direction.

It is straightforward to see that in choosing $\tilde{\varphi} = \tilde{f} = \tilde{g} = 1$, if the relation (9) is valid, then (17) is also valid. Therefore the above disorder solution has a one-dimensional behaviour in the horizontal direction.

For the particular case $\omega_1 = \omega_2$ and $\omega_3 = \omega_4$ the general 8-vertex model becomes the Baxter model, and the known condition $\omega_1 + \tilde{\omega}_7 = \omega_3 + \tilde{\omega}_5$ (Baxter 1982b) for the model to have a disorder solution is re-obtained from (14).

When $\omega_3 = \omega_4$ the condition (14) agrees with those found by Peschel and Rys (1982) and Rujan (1982) for the 8-vertex model with only one type of electric field, and the inequalities (15) become the inequality (4b) of the first of these papers.

For the case $\omega_1\omega_2\omega_3\omega_4 = \tilde{\omega}_5^2\tilde{\omega}_6^2$ the 8-vertex model reduces to the fully anisotropic square Ising model with diagonal interactions. Then, as a particular case, a disorder solution is obtained for this model from equations (14) and (16). For the isotropic case (equal horizontal and vertical interactions, and equal diagonal interactions) this solution agrees with those obtained by Peschel and Rys (1982). In general, all IRF models having diagonal and multispin interactions will have disorder solutions, which can be found in a very simple way by using the local condition (4).

To summarise: a disorder solution for the general 8-vertex model has been obtained. The partition function is given by (16) when the conditions (14) and (15) are satisfied. Moreover, the intra-row correlation function has a one-dimensional behaviour in the horizontal direction.

References

Baxter R J 1980 *Fundamental Problems In Statistical Mechanics V* (Amsterdam: North-Holland)
—— 1982a *J. Stat. Phys.* **28** 1
—— 1982b *Exactly Solved Models in Statistical Mechanics* (New York: Academic)
—— 1984 *J. Phys. A: Math. Gen.* **17** L911
Chao N C and Wu F Y 1985 *J. Phys. A: Math. Gen.* **18** L603
Fan C and Wu F Y 1970 *Phys. Rev.* B **2** 723
Hansel D, Georges A, Le Doussal P and Maillard J M 1985 *The vicinity of disorder varieties, preprint* A 674.0685
Jaekel M T and Maillard J M 1985a *J. Phys. A: Math. Gen.* **18** 641
—— 1985b *J. Phys. A: Math. Gen.* **18** 1229
—— 1985c *J. Phys. A: Math. Gen.* **18** 2271
Peschel I and Rys F 1982 *Phys. Lett.* **91A** 187
Rujan P 1982 *J. Stat. Phys.* **29** 231, 247
—— 1984 *J. Stat. Phys.* **34** 615
Wu F Y 1985 *J. Stat. Phys.* **40** 613